5.2

Downtown and Uptown

Graphs of Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Solve exponential functions using the intersection of graphs.
- Analyze asymptotes of exponential functions and their meanings in context.
- Identify the domain and range of exponential functions.
- Analyze and graph decreasing exponential functions.
- Compare graphs of linear and exponential functions through intercepts, asymptotes, and end behavior.

KEY TERM

- horizontal asymptote

PROBLEM 1 Downtown and Uptown

At this moment, the population of Downtown is 20,000, and the population of Uptown is 6000. But over many years, people have been moving away from Downtown at a rate of 1.5% every year. At the same time, Uptown's population has been growing at a rate of 1.8% each year.

1. What are the independent and dependent quantities in each situation?

   Indep: time (years) Dep: populations

2. Which city's population can be represented as an increasing function, and which can be represented as a decreasing function?

   Uptown: Increasing Function
   Downtown: Decreasing Function

Let's analyze the population growth of Uptown. In 1 year from now, the population of Uptown will be

$$6000 + 6000(0.018) = 6108.$$

The population will be 6108 people in Uptown 1 year from now.

3. Write and simplify an expression that represents the population of Uptown:

   a. 2 years from now.

   $6108 + 6108(0.018) = 6217.944$

   The population will be about 6218 people 2 years from now.

   b. 3 years from now.

   $6217.944 + 6217.944(0.018) = 6329.866992$

   The population will be about 6330 in 3 years from now.

You will have **3 minutes** to work in your group

4. How can you tell that this function is an exponential function? Explain your reasoning.

The rate of population increase for Uptown is not constant over time, but increases each year. (Every year the population goes up a little more than the year before.)

You can use the formula for compound interest to determine the function for Uptown's increasing population. Recall that the formula for compound interest is $P(t) = P(1 + r)^t$, where $P(t)$ represents the amount in the account after a certain amount of time in years, r is the interest rate written as a decimal, and t is the time in years.

5. In the compound interest formula, substitute Uptown's starting population for P and the rate of population growth for r.

   a. Write the function, $U(t)$, showing Uptown's population growth as a function of time in years.

   $U(t) = 6000(1+0.018)^t$ $\quad$ $U(t) = 6000(1.018)^t$

   b. Use your answers to Question 3 and a calculator to verify that your function is correct.

   $U(t) = 6000(1.018)^t$

   $U(t) = 6000(1.018)^2$ $\quad$ $U(t) = 6000(1.018)^3$

   $U(t) \approx 6218$ $\quad$ $U(t) \approx 6330$

Now let's analyze the population decline of Downtown.

6. Write and simplify an expression that represents the population of Downtown. The first one has been done for you.

   a. 1 year from now.

   $20{,}000 - 20{,}000(0.015) = 19{,}700$

   The population of Downtown will be 19,700 people 1 year from now.

   b. 2 years from now.

   $19{,}700 - 19{,}700(0.015) = 19{,}404.5$

   The population of Downtown will be about 19,405 people 2 years from now.

   c. 3 years from now.

   $19{,}404.5 - 19{,}404.5(0.015) = 19{,}113.4325$

   The population of Downtown will be about 19,113 people 3 years from now.

7. Rewrite the expressions for the population decline in Downtown using the Distributive Property. The first one has been done for you.

   a. 1 year from now.

      $20{,}000 - 20{,}000(0.015)$

      $20{,}000(1 - 0.015)$

   b. 2 years from now.

      $19{,}700 - 19{,}700(0.015)$

      $19{,}700(1 - 0.015)$

   c. 3 years from now.

      $19{,}404.5 - 19{,}404.5(0.015)$

      $19{,}404.5(1 - 0.015)$

8. Use the compound interest formula and your expressions in Question 7 to write the function, $D(t)$, showing Downtown's population decline as a function of time in years.

   $D(t) = 20{,}000(1 - 0.015)^t$

9. Think about each function as representing a sequence. What is the common ratio in simplest form, or the number that is multiplied each time to get the next term, in each sequence?

For Uptown, the common ratio is $1 + 0.018$, or 1.018. For Downtown, the common ratio is $1 - 0.015$, or 0.985.

10. Explain how the common ratios determine whether the exponential functions for the change in population are increasing or decreasing.

When the common ratio is greater than 1, the exponential function is increasing. When the common ratio is between 0 and 1, the exponential function is decreasing.

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PROBLEM 2 Graphing, Finally!

Let's examine the properties of the graphs of the functions for Downtown and Uptown. Here are the functions again:

Downtown: $D(t) = 20{,}000(1 - 0.015)^t$ Uptown: $U(t) = 6000(1 + 0.018)^t$

1. Use a graphing calculator to graph both functions using the bounds $[-100, 100] \times [0, 30{,}000]$.

2. Let's analyze the y-intercepts of each function.

   a. Identify the y-intercepts.

   Downtown: y-intercept is 20,000
   Uptown: y-intercept is 6000

   b. Interpret the meaning of the y-intercept in terms of this problem situation.

   It represents the current population of each city.

   c. Describe how you can determine the y-intercept of each function using just the formula for population increase or decrease.

   The y-intercept is the "starting" number of your function. The function is $f(x) = a \cdot b^x$

3. Use a graphing calculator to answer each question. Describe your strategy.

a. How long will it take for Downtown's population to be half of what it is now?

Between 45 and 46 years from now. You can graph a line at $y = 10{,}000$ and determine the x-value of the intersection point.

b. How long will it take for Uptown's population to double from what it is now?

Between 38 and 39 years from now. You can graph a line at $y = 12{,}000$ and determine the x-value of the intersection point.

c. How many years from now will the populations of Downtown and Uptown be equal? Determine the approximate populations.

Between 36 and 37 years from now. Just look at the x-value of the intersection point of the two functions.

Skip to problem 6.

Each population function you graphed has a *horizontal asymptote*. A **horizontal asymptote** is a horizontal line that a function gets closer and closer to, but never intersects.

6. Write the equation for the horizontal asymptote of each population function.

The equation for the horizontal asymptote is y = 0.

7. Does the horizontal asymptote make sense in terms of this problem situation? Explain your reasoning.

Not really. If a population is decreasing, it can eventually be "0." When you are strictly talking about numbers, though, it would get closer, but never reach zero.

8. Identify the domain and range of each function.

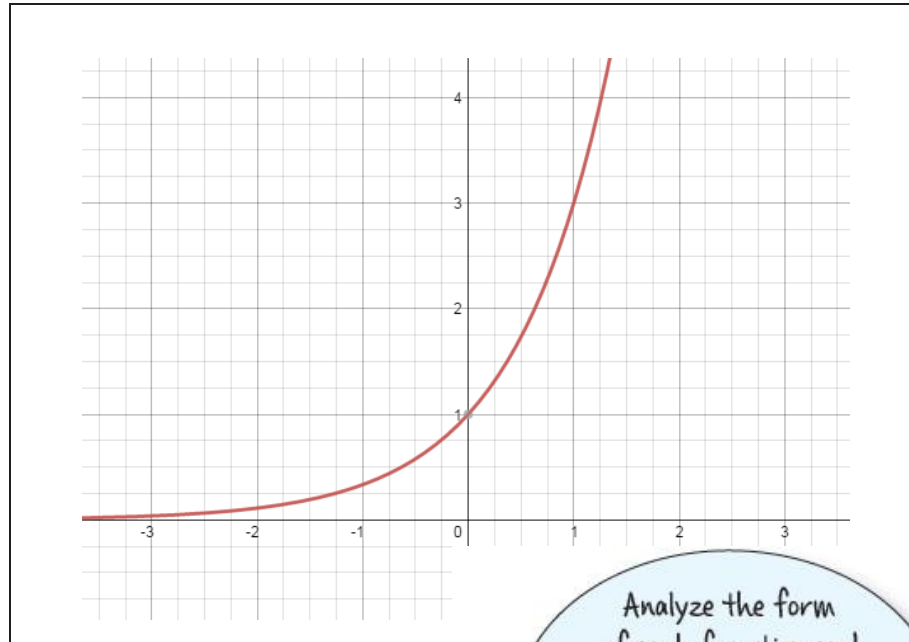
The domain is all real numbers (you can plug in any x-value). The range (y-values) is all real numbers greater than zero (y > 0).

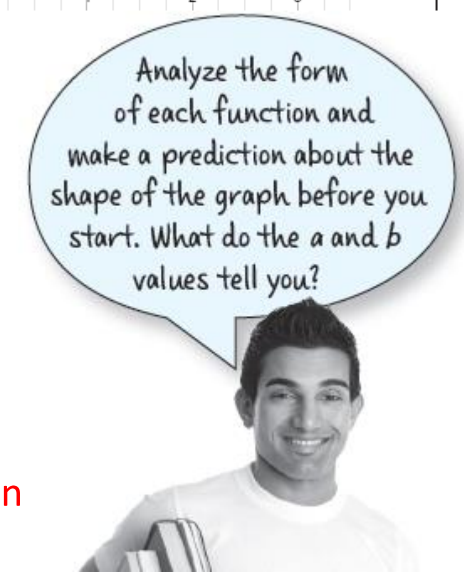
PROBLEM 3 The Multiple Representations of Exponentials

1. Complete the table and sketch a graph for each exponential function of the form $f(x) = ab^x$. Then determine the x-intercept(s), y-intercept, asymptote, domain, range, and interval(s) of increase/decrease.

a. $f(x) = 3^x$

x	$f(x)$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27





x-intercept(s): None

y-intercept: (0, 1)

asymptote: $y = 0$

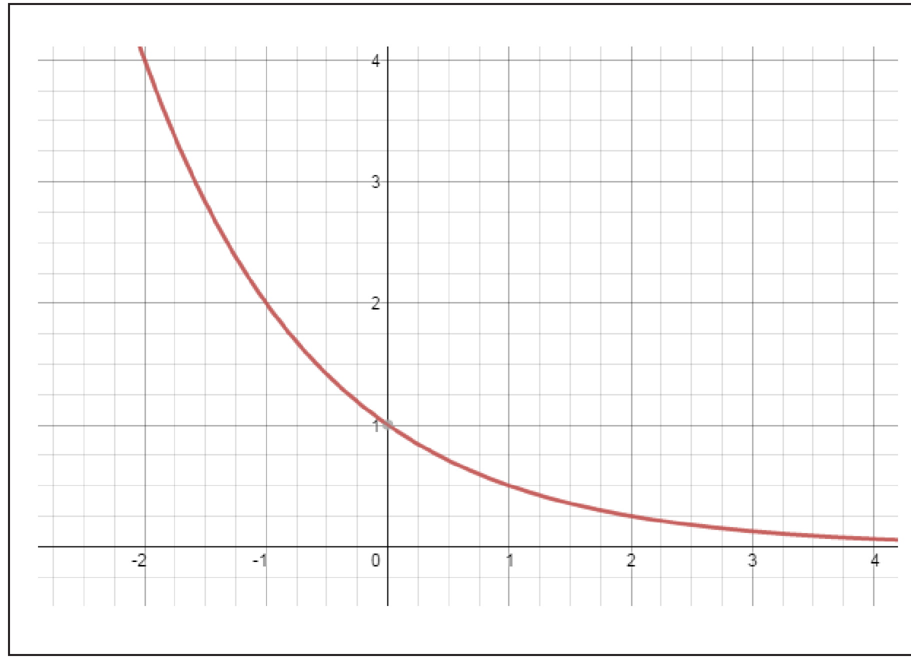
domain: All Real #'s

range: $y > 0$

interval(s) of increase/decrease: Increasing over the entire domain

b. $g(x) = \left(\frac{1}{2}\right)^x$

x	$g(x)$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



x-intercept(s): None

y-intercept: (0, 1)

asymptote: y = 0

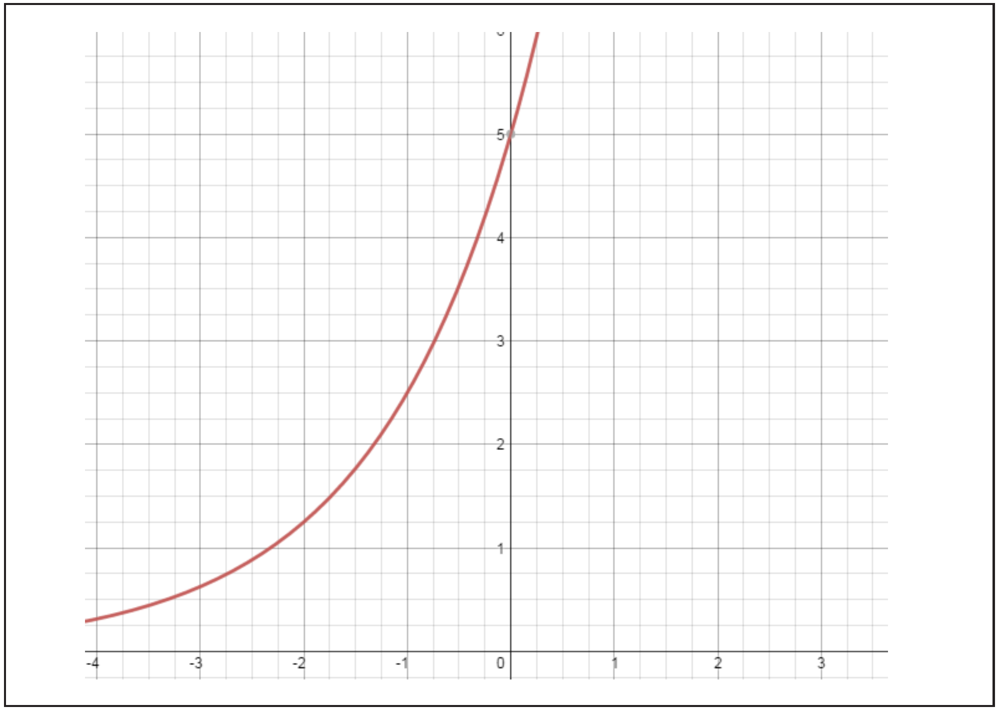
domain: All Real #'s

range: y > 0

interval(s) of increase/decrease: Decreasing over the entire domain

c. $k(x) = 5 \cdot 2^x$

x	$k(x)$
-2	$\frac{5}{4}$
-1	$\frac{5}{2}$
0	5
1	10
2	20
3	40



x-intercept(s): None

y-intercept: (0, 5)

asymptote: y = 0

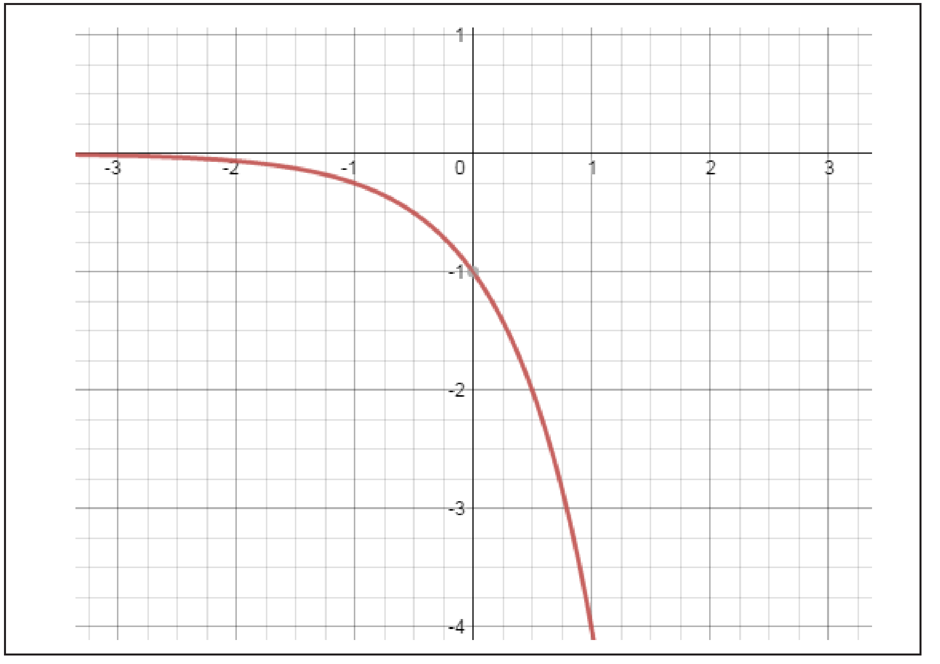
domain: All Real #'s

range: y > 0

interval(s) of increase/decrease: Increasing for entire domain

d. $p(x) = -4^x$

x	$p(x)$
-2	$-\frac{1}{16}$
-1	$-\frac{1}{4}$
0	-1
1	-4
2	-16
3	-64



x-intercept(s): None

y-intercept: (0, -1)

asymptote: y = 0

domain: All Real #'s

range: y < 0

interval(s) of increase/decrease: Decreasing for entire domain

2. Write an exponential equation of the form $y = ab^x$ for each. Explain your reasoning.

a.

x	y
0	1
1	4
2	16
3	64

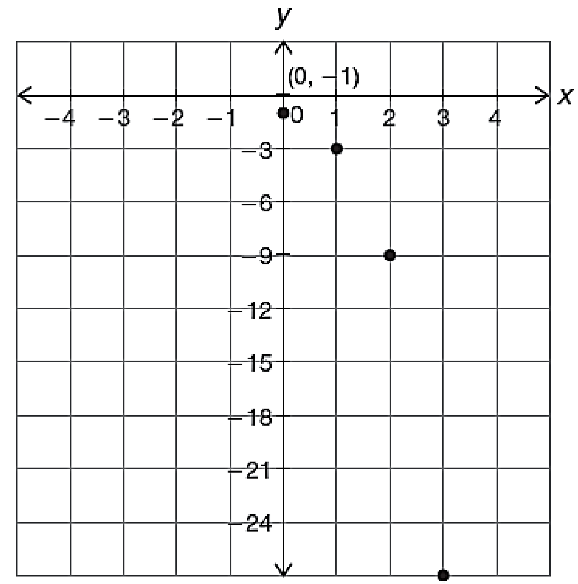
From the table we see the starting point (0,1)
So we know a = 1.

The common ratio is 4

Therefore the equation is $y = 1(4)^x$

$or \;\; y = 4^x$

b.



The y intercept is the starting point (0,-1) so a = -1

The common ratio is 3

Therefore the equation is $y = -1(3)^x$

$or \;\; y = -3^x$

3. Given a function of the form $f(x) = ab^x$.

a. What does the *a*-value tell you?

The y-intercept

b. What does the *b*-value tell you?

The common ratio between any two points